The collective model from a Cartan-Weyl perspective

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What's so geometrical about the geometrical model?

- Macroscopically, the atomic nucleus can be compared to a charged liquid drop.
- Deviations from the sphere are developed in multipole orders. Up to 2nd order

Radius

$$R(\theta, \phi) = R_0[1 + \alpha \cdot Y_2(\theta, \phi)]$$

- α^2_μ :: collective quadrupole[a] coordinates

[a] $L = 2$ tensor

A gallery of shapes

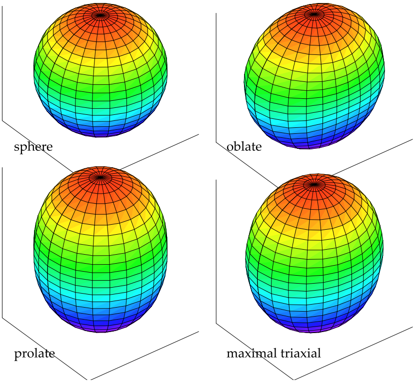
Radius

$$R(\theta,\phi) = R_0[1 + \alpha \cdot Y_2(\theta,\phi)]$$

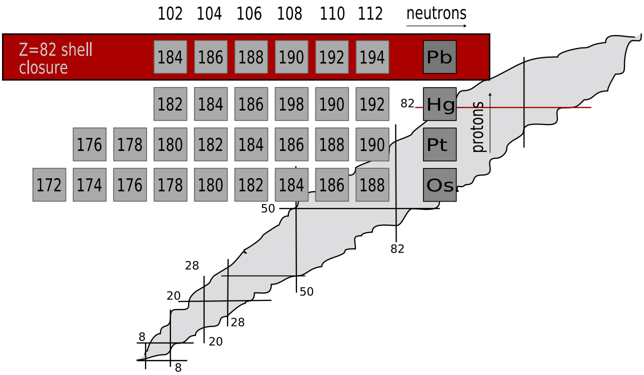
- Rotation to the intrinsic system

$$\begin{cases} \alpha'_0 = \beta \cos\gamma \\ \alpha'_2 = \alpha'_{-2} = \beta/\sqrt{2}\sin\gamma \\ \alpha'_1 = \alpha'_{-1} = 0 \end{cases}$$

- β is a measure for the deformation, γ for the triaxiality



The nuclear chart around Z=82 shell closure

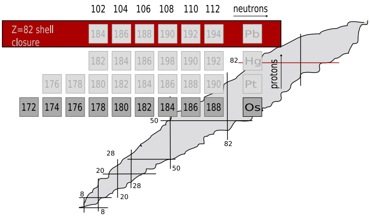


- Collective excitation modes are very important in the low-energy spectra
- Renewed interest due to the quantum shape phase transitions

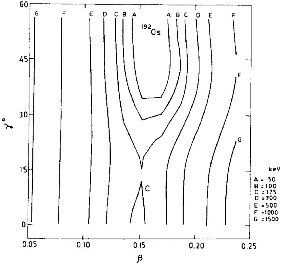
All kinds of shapes

neighbouring isotope chains

- Os :: triaxial nuclei
- Pt :: the γ-softie
- Pb :: 3 coexisting families



- Hartree-Fock mean field calculation
- A minimum can be found at $\gamma \neq 0$



A. Ansari, Phys. Rev. C 38 (1988) 953.

- Calculations in the framework of analytically solvable potentials

L. Fortunato et. al., Phys. Rev. C74 (2006) 014310.

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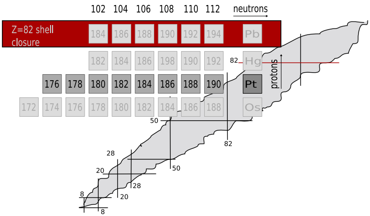
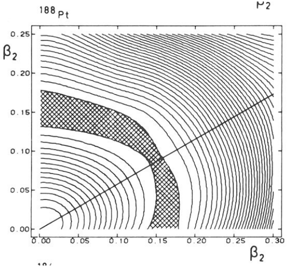


- Potential Energy Surface calculation
- Very flat γ dependence



R. Bengtsson et al., Phys. Lett. B 183 (1987) 1 (2004).

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neighbouring isotope chains

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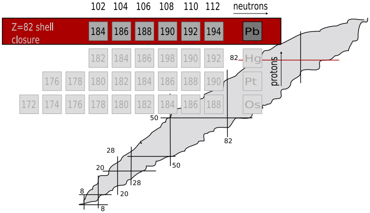
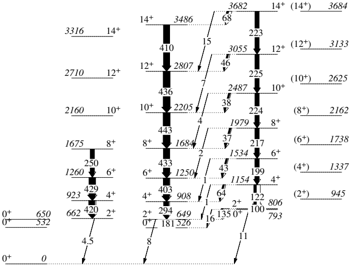
- **Pb :: 3 coexisting families**



- Interacting Boson Model calculation
- Extension to coexisting configurations points towards spherical-oblate-prolate structure



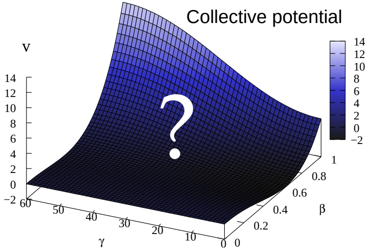
V. Hellemans, private communication (2007)

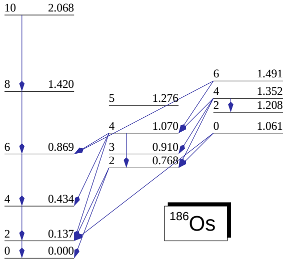
input/output of the geometrical model

The Bohr Hamiltonian

$$\hat{H} = \hat{T} + V(\alpha)$$

- kinetical egergy describes the stiffness of the surface
- $\hat{T} = \frac{1}{B_2}\pi \cdot \pi + B_3[\pi\alpha]^2 \cdot \pi + \dots$





- α describes small deformations
- Use a Taylor expansion

$$V(\alpha) = c_2(\alpha \cdot \alpha) + c_3([\alpha\alpha]^2 \cdot \alpha) + c_4(\alpha \cdot \alpha)^2 + c_5([\alpha\alpha]^2 \cdot \alpha)(\alpha \cdot \alpha) + \dots$$

1 The geometrical collective model
- The basics of the geometrical collective model
- The physics motivation

2 The collective variables in the Cartan-Weyl scheme
- An algebraic description...
- ... within the Cartan-Weyl scheme

3 Test application in quantum shape phase transitions

4 conclusions & outlook

The rules of the game

Commutation relations fix the structure

$$[\pi_\mu, \alpha_\nu] = -i\hbar\delta_{\mu\nu}, \quad [\alpha_\mu, \alpha_\nu] = 0, \quad [\pi_\mu, \pi_\nu] = 0$$

- The algebraic structure of the geometrical model is contained in the following recoupling formula

$$(\alpha \cdot \alpha)(\pi^* \cdot \pi^*) = (\alpha \cdot \pi^*)(\alpha \cdot \pi^*) + 3i\hbar(\alpha \cdot \pi^*) \\ - 2([\alpha\pi^*]^{(1)} \cdot [\alpha\pi^*]^{(1)} + [\alpha\pi^*]^{(3)} \cdot [\alpha\pi^*]^{(3)})$$

- It comprises the generators of the direct product group

$$\left.\begin{array}{l} \hat{Z}_1 = \alpha \cdot \alpha \\ \hat{Z}_2 = \pi^* \cdot \pi^* \\ \hat{Z}_3 = \alpha \cdot \pi^* \end{array}\right\} SU(1,1) \times O(5) \left\{ \begin{array}{l} \frac{i\hbar}{\sqrt{10}} L_M = [\alpha\pi^*]^{(1)}_M \\ \frac{i\hbar}{\sqrt{10}} O_M = [\alpha\pi^*]^{(3)}_M \end{array}\right.$$

Why $SU(1,1) \times O(5)$?

The Hamiltonian

$$\hat{H} = \frac{1}{2B_2}\pi \cdot \pi + B_3 \pi \cdot [\alpha\pi]^2 + c_2(\alpha \cdot \alpha) + c_3([\alpha\alpha]^2 \cdot \alpha) + c_4(\alpha \cdot \alpha)^2$$
$$+ c_5([\alpha\alpha]^2 \cdot \alpha)(\alpha \cdot \alpha) + c_6(\alpha \cdot \alpha)^3 + d_6([\alpha\alpha]^2 \cdot \alpha)^2 + \dots$$

$SU(1,1)$ basic block

$$\alpha \cdot \alpha = \beta^2$$

- The "radial" dependence
- Basis is known

$O(5)$ basic block

$$[\alpha\alpha]^2 \cdot \alpha = \sqrt{\frac{2}{7}}\beta^3 \cos 3\gamma$$

- The "angular" dependence
- Cartan-Weyl basis

The Cartan-Weyl basis of $O(5)$

Cartan's theorem

Every semi simple algebra of dimension n and rank r can be rotated to a natural basis for which

$$[H_i, H_j] = 0, \quad [H_i, E_\alpha] = \alpha_i E_\alpha, \qquad \{i,j\} \in r$$

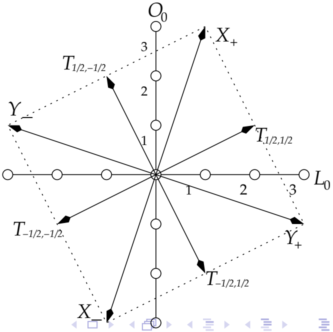
$$[E_\alpha, E_\beta] = N_{\alpha+\beta} E_{\alpha+\beta} \quad [E_\alpha, E_{-\alpha}] = \alpha^i H_i \qquad \{\alpha, \beta\} \in n - r$$

- Rotation
  $\{L_m, O_{m'}\} \to \{X_i, Y_j, T_{\mu\nu}\}$
- Group reduction is clear

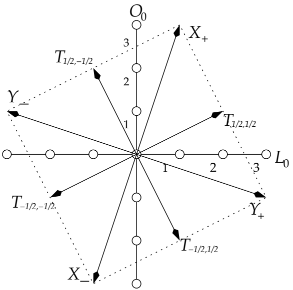
$$\underbrace{O(5)}_{v} \supset \underbrace{O(4)}_{X} \cong \underbrace{SU(2)}_{(X,M_X)} \times \underbrace{SU(2)}_{(X,M_Y)}$$

- A natural Cartan basis emerges

$$|vX(M_X, M_Y)\rangle$$



Action of the $O(5)$ generators on the natural basis (i)



- $\{X_\pm, X_0\}$ and $\{Y_\pm, Y_0\}$ span standard $SU(2)$ algebras
- According Racah: $T_{\mu,\nu}$ acts as a bispinor of character 1/2 in the $SU(2) \times SU(2)$ space

$$[X_0, T^{\frac{1}{2}\frac{1}{2}}_{\mu\nu}] = \mu T^{\frac{1}{2}\frac{1}{2}}_{\mu\nu}$$

$$[X_\pm, T^{\frac{1}{2}\frac{1}{2}}_{\mu\nu}] = \sqrt{(\tfrac{1}{2} \mp \mu)(\tfrac{1}{2} \pm \mu + 1)}\, T^{\frac{1}{2}\frac{1}{2}}_{\mu\pm 1\nu}$$

- The action of $T_{\mu,\nu}$ on a basis state $|vX(M_X, M_Y)\rangle$ is thus

$$T^{\frac{1}{2}\frac{1}{2}}_{\mu\nu}|vXM_XM_Y\rangle = a_+|v, X + \tfrac{1}{2}, M_X + \mu, M_Y + \nu\rangle + a_-|v, X - \tfrac{1}{2}, M_X + \mu, M_Y + \nu\rangle$$

- Applying the Wigner Eckart theorem twice

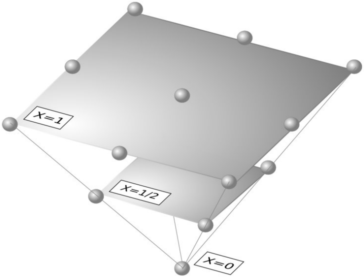
$$a_\pm = (-)^k \begin{pmatrix} X \pm \frac{1}{2} & \frac{1}{2} & X \\ -M_X - \mu & \mu & M_X \end{pmatrix} \begin{pmatrix} X \pm \frac{1}{2} & \frac{1}{2} & X \\ -M_Y - \nu & \nu & M_Y \end{pmatrix} \langle vX \pm \tfrac{1}{2} || T || vX\rangle$$

Action of the $O(5)$ generators on the natural basis (ii)

- For every v and X :: two unknown matrix elements two conditions needed

$$\mathcal{C}_2[O(5)] = 2(X^2 + Y^2 - 2[TT]^{(00)})$$
$$[T_{\mu\nu}, T_{\mu'\nu'}] = c_m^X X_0 + c_m^Y Y_0 \quad m = \{\mu\nu, \mu'\nu'\}$$

- norm selection rules $(X = 0 \ldots \frac{v}{2})$
- intermediate state method renders double reduced matrix elements



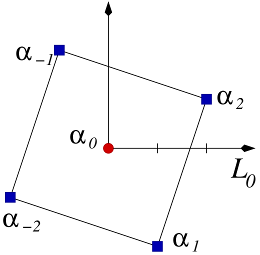
Example: Action of $T_{1/2,1/2}$

$$T_{\frac{1}{2}\frac{1}{2}}|vXM_XM_Y\rangle = \frac{\sqrt{(X+M_X+1)(X+M_Y+1)(v-2X)(v+2X+3)}}{2\sqrt{(2X+1)(2X+2)}}|vX+\tfrac{1}{2}M_X+\tfrac{1}{2}M_Y+\tfrac{1}{2}\rangle$$
$$-\frac{\sqrt{(X-M_X)(X-M_Y)(v-2X+1)(v+2X+2)}}{2\sqrt{(2X)(2X+1)}}|vX-\tfrac{1}{2}M_X+\tfrac{1}{2}M_Y+\tfrac{1}{2}\rangle$$

Tensor character of the α variables

basis block of the potential

$$\beta^3 \cos(3\gamma) \sim [\alpha\alpha]^{(2)} \cdot \alpha$$



- Need for matrix elements

$$\langle vX(M_X, M_Y)|\alpha_\mu|v'X'(M'_X, M'_Y)\rangle$$

- $\{\alpha_0\}$ and $\{\alpha_{-2}, \alpha_{-1}, \alpha_1, \alpha_2\}$ have bispinor 0 and $\frac{1}{2}$ character respectively

$$\alpha_0 \to \alpha^{00}_{00}, \qquad \{\alpha_{-2}, \alpha_{-1}, \alpha_1, \alpha_2\} \to \alpha^{\frac{1}{2}\frac{1}{2}}_{\mu\nu}$$

- Wigner Eckart

$$\langle vXM_XM_Y|\alpha^{\lambda\lambda}_{\mu\nu}|v'X'M'_XM'_Y\rangle = (-)^k \begin{pmatrix} X & \lambda & X' \\ -M_X & \mu & M'_X \end{pmatrix} \begin{pmatrix} X & \lambda & X' \\ -M_Y & \nu & M'_Y \end{pmatrix} \langle vX||\alpha^\lambda||v'X'\rangle$$

Closed expressions for the α matrix elements

What is used

$$[T_{\mu\nu}, \alpha^{\frac{1}{2}\frac{1}{2}}_{\mu'\nu'}] = \frac{(-)^{\mu-\nu}}{\sqrt{2}}\delta_{-\mu\mu'}\delta_{-\nu\nu'}\alpha^{00}_{00}$$

$$[T_{\mu\nu}, \alpha^{00}_{00}] = \frac{1}{\sqrt{2}}\alpha^{\frac{1}{2}\frac{1}{2}}_{\mu,\nu}$$

$$[\alpha^{\lambda\lambda}_{\mu,\nu}, \alpha^{\lambda'\lambda'}_{\mu'\nu'}] = 0$$

$$\alpha \cdot \alpha = \beta^2 = Z_1$$

- Seniority selection rules :: α is a $v = 1$ tensor
- Bispinor $\{\frac{1}{2}\frac{1}{2}\}$ can be expressed in terms of biscalar $\{00\}$ matrix elements
- Closed expressions for the matrix elements result

What is obtained: matrix elements of α

$$\langle vXM_XM_Y|\alpha^{00}_{00}|v+1, XM_XM_Y\rangle = \beta\sqrt{\frac{(v-2X+1)(v+2X+3)}{(2v+3)(2v+5)}}$$

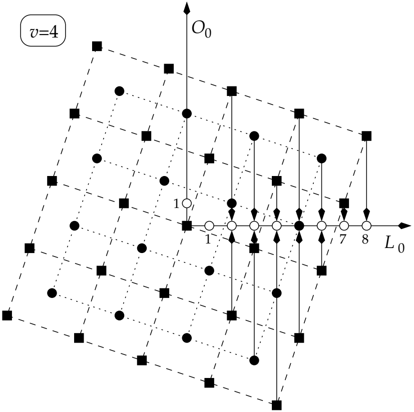
$$\langle vXM_XM_Y|\alpha^{00}_{00}|v-1, XM_XM_Y\rangle = \beta\sqrt{\frac{(v-2X)(v+2X+2)}{(2v+1)(2v+3)}}$$

Projection to the physical basis

- Experimental spectra have good angular momentum quantum number L
- The Hamiltonian is a scalar with respect to the angular momentum algebra $O(3)$

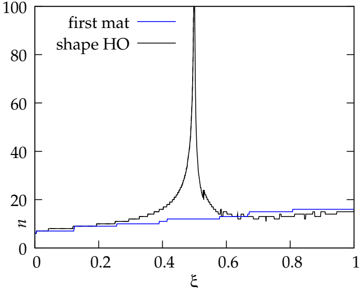
$$[L\cdot L,\mathcal{C}_G]\neq 0$$
$$[L_0,\mathcal{C}_G]=0$$

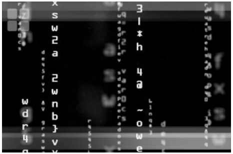
- Only the angular momentum projection is a good quantum number in the Cartan basis
- A rotation brings the natural- to the physical basis



All ingredients are ready

- α Matrix elements in $O(5)$ basis are derived
- Inclusion of $SU(1,1)$ basis is straightforward in a similar fashion
- Diagonalising = choosing a basis
- Harmonic oscillator = choosing $\hbar\omega$
- $H = \frac{1}{2B_2}\pi \cdot \pi + \xi V_{\text{vib}} + (1-\xi)V_{\gamma\text{-ind}}$
- Margetan & Williams Phys. Rev. C25 (1982) 1602





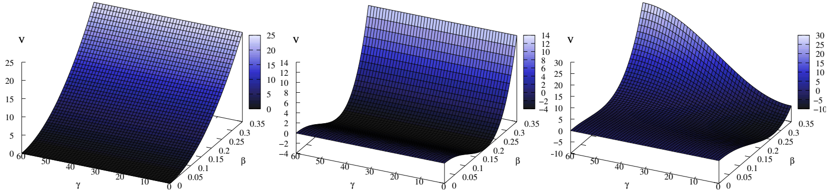
- Computer code is now under continuous development to diagonalise general collective potentials.
- Present status: upto β^4

Test application in quantum shape phase transitions

- Quantum shape phase transitions cover a large part of the model Hilbert space
- Ideal testground for the method
- Upto β^4, 3 meaningful limits result

3 limits

- vibrational limit
- γ-independent rotor
- axial deformed rotor



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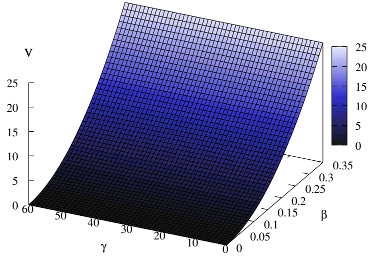
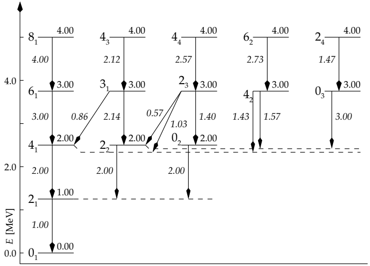


- Trivial limit
- Large degeneracies
- $B(E2)$ addition rule



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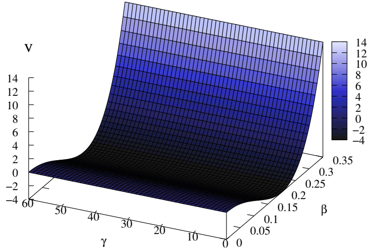
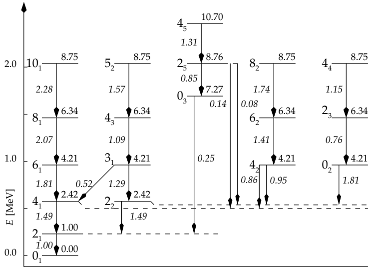


- Remaining seniority symmetry
- β-excitation band



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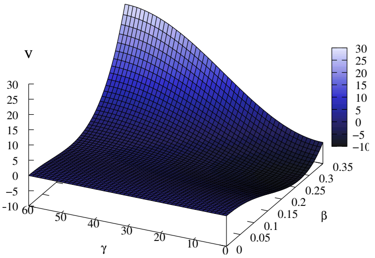
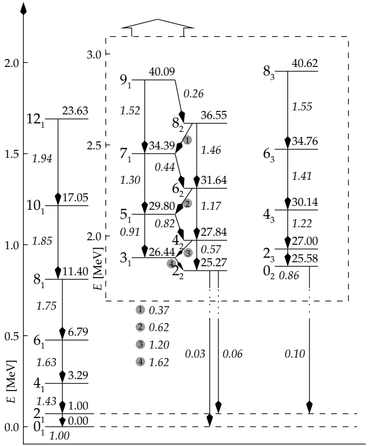
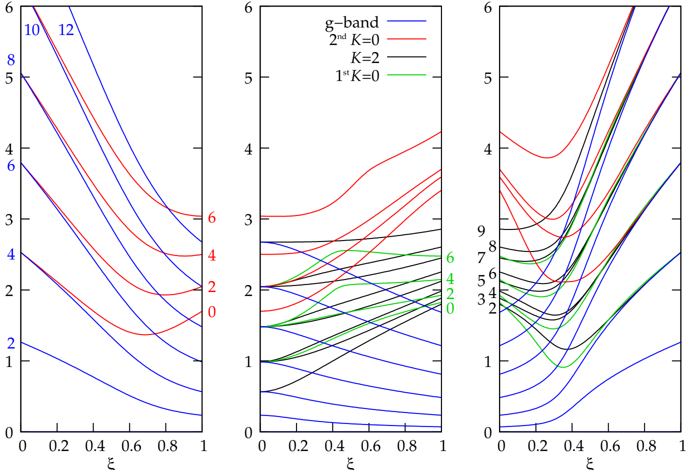


- Highly pronounced bands
- Rotation-Vibration model



Along the transition path: energy spectrum



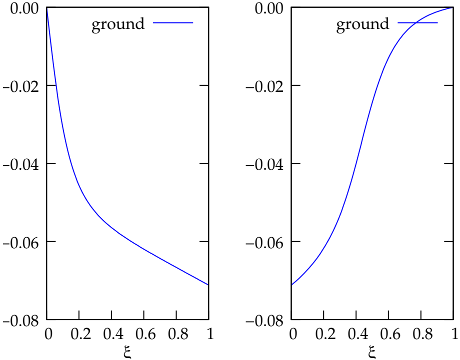
vibrator → γ-independent rotor → axial rotor → vibrator

Along the transition path: quadrupole moments

Quadrupole moments

$$Q = \langle \hat{Q}_\mu \rangle_{2_1} = \frac{3ZR_0^2}{4\pi} \langle \alpha_\mu \rangle_{2_1}$$

- α_μ is seniority $v = 1$ tensor
- $Q \equiv 0$ for vib. $\rightarrow$ γ-ind. rotor transition
- Other observables ($B(E2)$, $\rho(E0)$)



γ-ind. rotor $\rightarrow$ axial rotor $\rightarrow$ vibrator

Conclusions & outlook

- Collectivity accounts for a lot of the physics around $Z = 82$ shell closure
- All necessary matrix elements of a more general type of potential can be calculated in a Cartan-Weyl scheme
- First test applications in the framework of quantum shape phase transitions renders reliable results
- Further terms need to be included to study more general collective structures (e.g. triaxiality, shape coexistence), needed for the collectivity around the $Z = 82$ closed shell
- Possible extension to higher rank algebras ($O(7)$ octupole degrees of freedom and beyond)
- ...

Thank you for you attention!